QCD and hybrid NBD on oscillating moments of multiplicity distributions in lepton- and hadron-initiated reactions.

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Abstract

QCD predictions for moments of multiplicity distributions are compared with experimental data on e^+e^- collisions and their two-NBD fits. Moments of the multiplicity distribution in a two-NBD model for 1.8 TeV pp-collisions are considered. Three-NBD model predictions and fits for pp at LHC energies are also discussed. Analytic expressions for moments of hybrid NBD are derived and used to get insight into jet parameters and multicomponent structure of the processes. Interpretation of observed correlations is proposed.

Multiplicity distributions are the integral characteristics of multiparticle production processes. They can be described either in terms of probabilities $P_n(E)$ to create n particles at energy E or by the moments of these distributions. It has been found that their shapes possess some common features in all reactions studied. At comparatively low energies below tens of GeV, these distributions are relatively narrow and have sub-Poissonian shapes. At energies about 20 GeV for e^+e^-- annihilation and 30 GeV for pp (and $p\bar{p}$) interactions, they can be well fitted by the Poisson distribution[2]. At higher energies, the shapes become super-Poissonian, i.e. their widths are larger than for Poisson distribution. They increase with energy and, moreover, some shoulder-like substructures appear.

Their origin is usually ascribed to multicomponent contents of the process. In QCD description of e^+e^--processes these could be subjets formed inside quark and gluon jets (for the reviews see, e.g., [1, 2]). In phenomenological approaches, the multiplicity distribution in a single subjet is sometimes approximated by the negative binomial distribution (NBD) first proposed for hadronic reactions in [3]. For hadron-initiated processes, these peculiarities are also explained by the multicomponent structure of the process. This is either multiladder exchange in the dual parton model [4, 5], varying number of clans [6] or multiparton interactions [7, 8]. These subprocesses are related to the matter state during the collision (e.g., there are speculations about nonhomogeneous matter distribution in

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[2]This difference of thresholds can be attributed to lower energy spent in hadron collisions for new particles creation (due to the so-called leading particle effect) compared to e^+e^-- annihilation.

impact parameters [9], not to speak of quark-gluon plasma [10] behaving as a liquid [11] etc).

Such evolution of the multiplicity distributions can be quantitatively described by the energy behaviour of their moments. These moments reveal the correlations inherent for the matter state formed during the collision. Similarly to virial coefficients in statistical physics they can tell us about the equation of state of this matter. To introduce them, let us write the generating function of the multiplicity distribution as

$$G(E, z) = \sum_{n=0}^{\infty} P_n(E)(1 + z)^n. \tag{1}$$

In what follows, we will use the so-called unnormalized factorial $\mathcal{F}_q$ and cumulant $\mathcal{K}_q$ moments defined according to the formulae

$$\mathcal{F}_q = \sum_n P_n n(n - 1)...(n - q + 1) = \left.\frac{d^q G(E, z)}{dz^q}\right|_{z=0}, \tag{2}$$

$$\mathcal{K}_q = \left.\frac{d^q \ln G(E, z)}{dz^q}\right|_{z=0}. \tag{3}$$

They correspondingly define the total and genuine correlations among the particles produced (for more details see [12, 2]). These cumulant moments could be considered as the direct analogies of virial coefficients of statistical physics since both are related to genuine (irreducible) correlations. In particular, the first moments describe the mean multiplicity $\langle n \rangle$

$$\mathcal{F}_1 = \mathcal{K}_1 = \langle n \rangle, \tag{4}$$

and the second moments are related to the dispersion D of the distribution P_n:

$$\mathcal{K}_2 = \mathcal{F}_2 - \langle n \rangle^2 = D^2 - \langle n \rangle. \tag{5}$$

The higher rank moments reveal other asymmetries of distributions such as skewness etc. Since both $\mathcal{F}_q$ and $\mathcal{K}_q$ strongly increase with their rank and energy, their ratio

$$H_q = \mathcal{K}_q / \mathcal{F}_q \tag{6}$$

first introduced in [13] is especially useful due to partial cancellation of these dependences. The factorial moments $\mathcal{F}_q$'s are always positive by definition (Eq. (2)) while the cumulant moments $\mathcal{K}_q$'s can change sign. Again, let us recall that the changing sign second virial coefficient in statistical physics implies the liquid state with the Van der Waals equation

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corresponding to repulsion at small distances and attraction at large distances. Cooper pair formation is also related to similar behaviour of correlations.

Here, we compare QCD and NBD approaches to the description of multiplicity distributions. We argue that H_q values are more sensitive to minute details of the distributions than their direct chi-square fits and reveal differences between proposed fits of e^+e^- and pp ($p\bar{p}$) processes. Some estimates for LHC energies will be provided.

The generating functions for quark and gluon jets satisfy definite equations in perturbative QCD (see [14, 2]). It has been analytically predicted in gluodynamics [13] that at asymptotically high energies the H_q moments are positive and decrease as q^{-2} but at present energies they become negative at some values of q and reveal the negative minimum at

$$q_{min} = \frac{1}{h_1\gamma_0} + 0.5 + O(\gamma_0), \tag{7}$$

where $h_1 = b/8N_c = 11/24$, $b = 11N_c/3 - 2n_f/3$, $\gamma_0^2 = 2N_c\alpha_S/\pi$, α_S is a coupling strength, N_c, n_f are the numbers of colours and flavours. At Z^0 energy $\alpha_S \approx 0.12$, and this minimum is at about $q \approx 5$. It moves to higher ranks with energy increase because the coupling strength decreases. Some hints to possible oscillations of H_q vs q at higher ranks at LEP energies were obtained in [13]. Then the approximate solution of the gluodynamics equation for the generating function [15] agreed with this and predicted the oscillating behaviour at higher ranks. These oscillations were confirmed by experimental data for e^+e^- and hadron-initiated processes first in [16], later in [17] and most recently in [18]. The same conclusions were obtained from exact solution of equations for quark and gluon jets in the framework of fixed coupling QCD [19]. The physics interpretation of these oscillations as originating from multisubjet structure of the process is related to the (multi)fractal behaviour of factorial moments, found also in QCD [20, 21, 22]. The asymptotic disappearance of oscillations can be ascribed to the extremely large number of subjets at very high energies.

A recent exact numerical solution of the gluodynamics equation in a wide energy interval [23] coincides with the qualitative features of multiplicity distributions described above. In terms of moments they correspond to the values of H_q changing sign at each subsequent q (with $H_2 < 0$) at low energies (narrow shapes[3]), the approach of H_q to zero at the Poisson transition point about 20 GeV for e^+e^- processes, and the positive second moment H_2 with oscillations of higher rank cumulants at Z^0 which disappear asymptotically. At Z^0, the first minimum appears at $q \approx 5$. This confirms earlier exact QCD results [24] at Z^0. It moves to higher ranks with a steadily decreasing amplitude

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[3]Narrow distributions always have such cumulants as shown, e.g., in [2].

when energy increases. The only free parameter is the QCD cut-off, which is however approximately fixed by the coupling strength and does not strongly influence the results.

In parallel, the NBD-fits of multiplicity distributions were attempted [6, 25]. The single NBD-parameterization is

$$P_n(E) = \frac{\Gamma(n + k_1)}{\Gamma(n + 1)\Gamma(k_1)} \left(\frac{n_1}{k_1}\right)^n \left(1 + \frac{n_1}{k_1}\right)^{-n - k_1}, \tag{8}$$

where Γ denotes the gamma-function. This distribution has two adjustable parameters $n_1(E)$ and $k_1(E)$ which depend on energy. Such a formula happened to describe low energy data with negative values of k_1 that corresponds to binomial fits. At the Poisson transition point $k^{-1} = 0$. The parameter k_1 becomes positive at higher energies. However the simple fit by the formula (8) is valid till the shoulders appear. In that case, this formula is replaced by the hybrid NBD which combines two or more expressions like (8). Each of them has its own energy dependent parameters n_i, k_i. These distributions are weighted with the energy dependent probability factors α_i which sum up to 1. Correspondingly, the number of adjustable parameters drastically increases.

A single NBD (8) has positive cumulants for $k_1 > 0$ ($\mathcal{K}_q = \Gamma(q)n_1^q/k_1^{q-1}$) and thus the positive $H_q = \Gamma(q)\Gamma(k_1 + 1)/\Gamma(k_1 + q)$. For hybrid NBD, the negative H_q can exist. The traditional procedure to calculate higher rank moments is by the iterative relations

$$H_q = 1 - \sum_{m=1}^{q-1} \frac{\Gamma(q)}{\Gamma(m + 1)\Gamma(q - m)} H_{q-m} \frac{\mathcal{F}_m \mathcal{F}_{q-m}}{\mathcal{F}_q}. \tag{9}$$

The strong compensations are inherent in Eq. (9). This calls for high accuracy of numerical calculations. More important, the formula does not give any direct insight into the physical reasons for such compensations. Therefore, it is instructive to write the analytic formulae for moments of hybrid NBD which provide clear interpretation of negative values of cumulants. We have derived these expressions for the two-NBD parameterization (2NBD) given by a sum of two expressions like (8) with two sets of adjustable parameters $n_1, k_1, \ n_2, k_2$ weighted with energy dependent factors α and $1 - \alpha$ correspondingly. 2NBD describes the process with two independent NBD-components of mean multiplicities n_i and widths k_i created with probabilities α and $1 - \alpha$. The factorial moments for any rank q are given by the simple formula

$$\mathcal{F}_q = \alpha \frac{\Gamma(k_1 + q)}{\Gamma(k_1)} \frac{n_1^q}{k_1^q} + (1 - \alpha) \frac{\Gamma(k_2 + q)}{\Gamma(k_2)} \frac{n_2^q}{k_2^q} \qquad (0 \le \alpha \le 1). \tag{10}$$

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The cumulant moments are more complicated and should be calculated separately for each rank. The first 5 moments are

$$\mathcal{K}_1 = \mathcal{F}_1 = \langle n \rangle = \alpha n_1 + (1-\alpha)n_2, \tag{11}$$

$$\mathcal{K}_2 = \frac{\alpha n_1^2}{k_1} + \frac{(1-\alpha)n_2^2}{k_2} + \alpha(1-\alpha)(n_1 - n_2)^2, \tag{12}$$

$$\mathcal{K}_3 = \frac{2\alpha n_1^3}{k_1^2} + \frac{2(1-\alpha)n_2^3}{k_2^2} + \alpha(1-\alpha)(n_1 - n_2)[3(n_1^2/k_1 - n_2^2/k_2) + (1-2\alpha)(n_1 - n_2)^2], \tag{13}$$

$$\mathcal{K}_4 = \frac{6\alpha n_1^4}{k_1^3} + \frac{6(1-\alpha)n_2^4}{k_2^3} + \alpha(1-\alpha)[(n_1 - n_2)^4(1 - 6\alpha(1-\alpha)) + 11(n_1^2/k_1 - n_2^2/k_2) -$$
$$8n_1 n_2(n_1/k_1 - n_2/k_2)^2 + 6(1-2\alpha)(n_1 - n_2)^2(n_1^2/k_1 - n_2^2/k_2)], \tag{14}$$

$$\mathcal{K}_5 = \frac{24\alpha n_1^5}{k_1^4} + \frac{24(1-\alpha)n_2^5}{k_2^4} + 5\alpha(1-\alpha)[6(n_1 - n_2)(n_1^4/k_1^3 - n_2^4/k_2^3) +$$
$$4(n_1^3/k_1^2 - n_2^3/k_2^2)(n_1^2/k_1 - n_2^2/k_2) + (1-2\alpha)(n_1 - n_2)(7(n_1 - n_2)(n_1^3/k_1^2 - n_2^3/k_2^2) +$$
$$3n_1 n_2(n_1/k_1 - n_2/k_2)^2) + 2(1 - 6\alpha(1-\alpha))(n_1 - n_2)^3(n_1^2/k_1 - n_2^2/k_2) +$$
$$0.2(1-2\alpha)(1 - 12\alpha(1-\alpha))(n_1 - n_2)^5], \tag{15}$$

For $\alpha = 0$ or 1 they reduce to one-NBD formulae with one of the first two terms surviving. It is always positive for positive k_i. Therefore, as expected, individually considered, the distributions show no oscillations. For 2NBD, there is a symmetry in replacing indices 1 to 2 together with α to $1-\alpha$. Negative $\mathcal{K}_2$ can be obtained only if $k_i < 0$. For positive k_i one always gets positive $\mathcal{K}_2$. Its value depends on the difference $n_1 - n_2$. $\mathcal{K}_3$ can become negative depending on the values of the last two terms. The cancellations in the expressions (13)-(15) are not so drastic as in Eq. (9), especially for large q, because the leading contributions to H_q are strongly decreasing with q there and not of the order of 1 as in (9). Therefore they do not require very high precision and, moreover, clearly display the origin of each term and its dependence on fitted parameters.

Actually, five moments determine quite well the shape of the distribution if they are calculated with high enough accuracy. Since these shapes are qualitatively similar in different reactions, it is especially instructive to compare their H_q moments. In Table 1 the H_q moments for e^+e^- annihilation at Z^0 are shown. Their values according to the solution of the gluodynamics equations [23] are in the first column. In the second and third columns, the experimental results of L3 collaboration [18] are represented for full phase space (FPS), correspondingly, with all measured multiplicities included and with some very high multiplicities truncated (because of large error bars). Next follow H_q values restored from 2NBD fits of OPAL and DELPHI results done in [26].

Table 1.

	QCD	L3, untr	L3, tr	OPAL,2NBD	DELPHI,2NBD
H_2	3.9E-2	(4.42±0.11)E-2	(4.41±0.10)E-2	4.4E-2	3.1E-2
H_3	7.4E-3	(7.40±0.38)E-3	(7.20±0.35)E-3	7.4E-3	2.6E-3
H_4	4.0E-4	(9.69±2.56)E-4	(7.17±1.42)E-4	4.9E-4	7.4E-5
H_5	-2.2E-4	-(1.30±1.59)E-4	-(3.95±0.53)E-4	-2.4E-4	-7.3E-5

The overall agreement is rather good. Quite impressive is the fact that in all cases the fifth cumulant moment is negative. However, somewhat surprising is the difference of the theoretical and experimental widths (H_2 values). The widths are determined quite precisely both experimentally and theoretically. The only reason to which such disagreement could be ascribed is the incomplete treatment with quarks omitted in [23]. The more complete approach will shed more light on this problem. The even stronger disagreement with DELPHI data is probably related to the special selection of events there. Further analysis is needed.

The comparison of e^+e^- and pp ($p\bar{p}$) data turns out especially interesting. While both show qualitative similarity of the shapes of multiplicity distributions, the corresponding H_q values are quite distinctive. In Table 2 we show H_q values for $p\bar{p}$ at 1.8 TeV (Tevatron) and interpolations to 14 TeV (LHC). The 2NBD fit at 1.8 TeV corresponds to the following parameters: $\alpha = 0.62$, $n_1 = 30$, $n_2 = 61.6$, $k_1 = k_2 = 7$, which are approximately equal to average values for 2.A model considered in [6]. However, even this extreme model underestimates high multiplicities and, therefore, H_q values in the Table should be treated as lower bounds to experimental ones, which are unknown, unfortunately. The extrapolated values at 14 TeV have been calculated using the parameters of 3NBD fits and Pythia model considered in [25].

Table 2.

	2NBD fit, 1.8TeV	3NBD fit, 14TeV	Pythia, 14TeV
H_2	0.2279	0.8754	0.4224
H_3	0.0988	0.9703	0.3387
H_4	0.0414	0.9737	0.2683
H_5	0.0120	0.9742	0.1877

Quite impressive are much larger values of H_q in hadron-initiated reactions (Table 2) as compared to e^+e^- results (Table 1). They strongly increase with energy. Moreover,

the drastic difference is clearly displayed by H_q between 3NBD interpolations and Pythia at 14 TeV. This demonstrates extremely high sensitivity of H_q analysis because both approaches provide the similar two-shoulder structure of multiplicity distributions as seen in Fig. 2 of [6]. At 14 TeV, the predictions are given for full phase space. For the rapidity interval $|\eta| < 0.9$ the H_q values become larger than those in Table 2. H_q for the 3NBD-model of [6] become almost indistinguishable from 1 (above 0.99). Pythia values increase about 1.4 times. No oscillations are seen at these high energies while they are present at energies below 1 TeV [16]. Surely, LHC experiments will give their decisive conclusion.

To conclude, we have shown that H_q moments of the multiplicity distribution are extremely sensitive to minute details of its shape. They can resolve the differences between various fits even if those are not clearly seen in the traditional representation. For e^+e^-, slight disagreement on theoretical and experimental widths is embarassing and must be further studied. For hadron- and nuclei-initiated reactions, H_q values are much larger than in e^+e^-. RHIC and LHC data are awaited for better insight. The energy dependence of H_q and of the relative weights of various NBD components can provide some hints on the matter state during the collision and its energy evolution.

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